An Exact Parallel Algorithm for Traveling Salesman Problem

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Definitions

Hamiltonian Cycle

A Hamiltonian cycle is a graph cycle that visits each node exactly once.

Traveling Salesman Problem

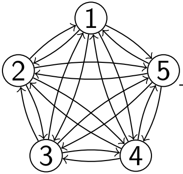
The traveling salesman problem is a problem of finding a minimum Hamiltonian cycle on a complete oriented graph with non-negative edge costs. It is NP-hard.

Balas' and Christofides' Algorithm

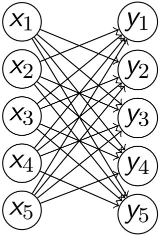
- Exact;
- Branch-and-bound;
- Each branch-and-bound tree node has up to $\frac{n}{2}$ branches (n – number of nodes in the graph);
- Branch-and-bound tree usually has small height;
- Works on any complete graph with non-negative edge costs.

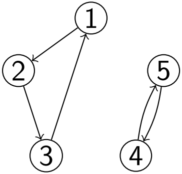
The Core Idea

- The Hungarian algorithm is used to solve the associated assignment problem;



$$\begin{array}{c|c} & \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} & \begin{pmatrix} \infty & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & \infty & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & \infty & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & \infty & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{54} & \infty \end{pmatrix} \end{array}$$





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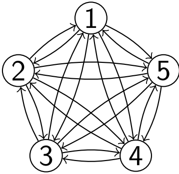

$$
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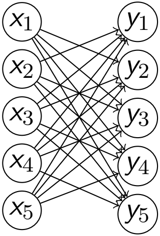

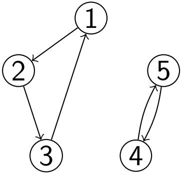

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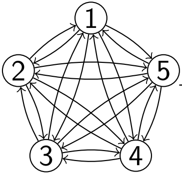

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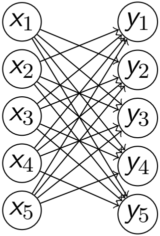

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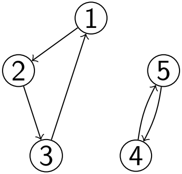

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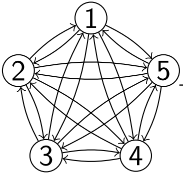

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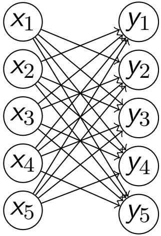

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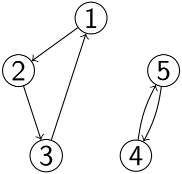

- Its solution forms either a Hamiltonian cycle (then the cycle is optimal on the current branch), or a union of disjoint simple subcycles on the initial graph, and said union covers all nodes of the graph;
- We merge the subcycles until we obtain a Hamiltonian cycle. The cycle is optimal on the current branch;
- Different branches consider different ways of merging the subcycles.

Our Modifications

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- ✓ We parallelized the algorithm with OpenMP;

- ✗ Balas' and Christofides' bounding procedures and the second branching method are excluded.

Implementation Details and Test Environment

- Programming language: C++;
- Compiler: GCC v. 6.3.0, compiler option: -Ofast;
- GNU OpenMP v. 6.3.0;
- OS: Debian 9 (Linux);
- Processor: Intel® Core™ i5-6600 CPU @ 3.30GHz
  - 4 cores;
  - no hyperthreading;
  - L3: 6 MB (shared);
  - L2: 256 kB (split);
  - L1: 32 kB instruction cache, 32 kB data cache (split);
- RAM: DDR4, 32 GB, clock speed: 2133 MHz;
- Download link:
  http://ops.rsu.ru/download/progs/BalasChristofides_v1_0.zip

Results

(On graphs with uniform random integer edge costs in range from 0 to 1 000 000)

Number of nodes	Average Time, sec		Speedup
	Sequential	Parallel (4 Cores)	
1000	4.3838	3.7466	1.17
1500	12.5135	11.9873	1.04
2000	60.5096	32.5627	1.86
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- During a sequential tree traversal the current best value of the cost function can (and probably will) improve several times, which helps to cut unnecessary branches closer to the root;
- Threads of a parallel program could explore the unneeded branches further than one thread of a sequential program would, because the current best value could not have been improved enough to eliminate them.

Different Tree Traversal Order (TTO)

Number of nodes	Average Time, sec (Parallel Algorithm, 4 Cores)	
	Balas' and Christofides' TTO[1]	Our TTO
1000	7.1217	3.7466
1500	29.0514	11.9873
2000	61.9874	32.5627
2500	137.3588	50.7420
3000	219.6173	75.8543

[1] Their TTO was used in our algorithm

Comparison to Other Algorithms

Number of nodes	Average Time, sec		
	Concorde	Fischetti-T.	Our Parallel
1000	3954.1[1]	4.6[1]	3.7

[1] Results from http://www.graphalgorithms.it/erice2008/Talks/ATSP_Lecture_Erice_Toth.pdf

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Fischetti-T. and Concorde were run on matrices with smaller edge cost range (from 0 to 1 000), which is not a good choice for such problem sizes.

Number of nodes	Average Time, sec (Parallel Algorithm, 4 Cores)	
	0...1 000 000	0...1 000
1000	3.7466	1.0018
1500	11.9873	1.0836
2000	32.5627	1.1875
2500	50.7420	2.1314
3000	75.8543	1.8445

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Further Improvements

- Decrease memory consumption;

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- Decrease memory consumption;
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- Experiment on sparse graphs.

Thank you!
Any questions?